AXISYMMETRIC DEFORMATION AND TORSION OF A TRANSVERSELY ISOTROPIC CYLINDER UNDER THE ACTION OF A POLYNOMIAL LOAD

(OSESIMMETRICHNAIA DEFORMATSIIA I KRUCHENIE TRANSVERSAL'NO IZOTROPNOGO TSILINDRA POD DEISTVIEM POLINOMIAL'NOI NAGRUZKI)

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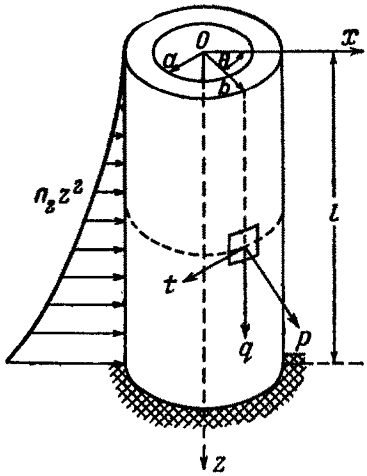
The paper considers the problem of the elastic equilibrium of a transversely isotropic cylinder under the action of forces distributed over its lateral surface according to an integer polynomial in the distance from the end of the cylinder, the forces being independent of the polar angle θ. A number of authors have studied various variants of the problem of the stress distribution in an isotropic cylinder under the action of polynomial loading (Almansi's problem); the method of solution for the axisymmetric case and the appropriate references are indicated in Chapter 7 of [1]. The present paper gives a general method of solution based on the application of the theory of axisymmetric deformation and torsion similar to the method suggested by Lur'e for the solution of the problem of an elastic layer; it enables the conditions on the cylindrical surfaces to be satisfied exactly, and on the ends approximately, "on the average".

1. General expressions for stresses and displacements. Let us consider an elastic body in the form of a hollow circular cylinder of finite length possessing transverse isotropy, i.e. at every point in the cylinder there is a plane for which all directions are elastically equivalent; we shall assume that this plane is normal to the axis of the cylinder. We shall refer the body to a system of cylindrical co-ordinates r, θ, z, with axes as shown in the diagram.

Let us suppose that the cylinder is subjected to pressures p, q, t and p', q', t', which act on the external and internal surfaces, respectively (in radial, axial and tangential directions), and which are independent of the polar angle θ.

We shall assume that the material follows the generalized Hooke's law, and that the induced strains are small. Using the conventional notations



for stresses and strains, we write the equations of the generalized Hooke's law as follows [2]:

$$\begin{aligned} \varepsilon_r &= \frac{1}{E}(\sigma_r - \nu\sigma_\theta) - \frac{\nu_1}{E_1}\sigma_z, & \gamma_{rz} &= \frac{1}{G_1}\tau_{rz} \\ \varepsilon_\theta &= \frac{1}{E}(-\nu\sigma_r + \sigma_\theta) - \frac{\nu_1}{E_1}\sigma_z, & \gamma_{\theta z} &= \frac{1}{G_1}\tau_{\theta z} \\ \varepsilon_z &= -\frac{\nu_2}{E}(\sigma_r + \sigma_\theta) + \frac{1}{E_1}\sigma_z, & \gamma_{r\theta} &= \frac{1}{G}\tau_{r\theta} \end{aligned} \tag{1.1}$$

Here, E, E_1 are the Young's moduli for tension-compression in the plane of isotropy and in a direction normal to this plane; ν, ν_1, ν_2, are Poisson's ratios; G, G_1 are the shear moduli for the plane of isotropy and for radial planes, so that

$$E\nu_1 = E_1\nu_2, \qquad G = \frac{E}{2(1+\nu)} \tag{1.2}$$

We introduce the following notations: a, b denote the internal and external radii and l the length of the cylinder

$$H = E\nu_1 + G_1(1 - \nu - 2\nu_1\nu_2)$$

$$\alpha = 2G\frac{G_1(1-\nu_1\nu_2)}{H}, \qquad \beta = G_1\frac{E\nu_1}{H}$$

$$\alpha_1 = 2G\frac{E_1 - G_1\nu_1(1+\nu)}{H}, \qquad \beta_1 = G_1\frac{E_1(1-\nu)}{H}$$

$$\gamma = 2G\frac{1-\nu_1\nu_2}{H}, \qquad \delta = G_1\frac{1-\nu-2\nu_1\nu_2}{H} \tag{1.3}$$

$$s_0 = \sqrt{\frac{G}{G_1}}, \qquad s_{1,2} = \sqrt{\frac{\alpha_1-\beta\pm\sqrt{(\alpha_1-\beta)^2-4\alpha\beta_1}}{2\beta_1}}$$

$$D^2 = \frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right), \qquad D_1^2 = \frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr} = r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)$$

Here D^2 is the Laplace operator for a function dependent on r only.

In all cases when the stresses and displacements are independent of θ they can be expressed in terms of two functions $F(r, z)$ and $\phi(r, z)$; i.e.

$$u_r = -\frac{\partial^2 F}{\partial r\,\partial z}, \qquad u_\theta = \frac{\partial\varphi}{\partial r}, \qquad w = \gamma D^2F + \delta\frac{\partial^2 F}{\partial z^2} \tag{1.4}$$

$$\begin{aligned} \sigma_r &= \frac{\partial}{\partial z}\left(2G\frac{1}{r}\frac{\partial F}{\partial r}-\alpha D^2F+\beta\frac{\partial^2F}{\partial z^2}\right), \qquad \tau_{rz} = \frac{\partial}{\partial r}\left(\alpha D^2F-\beta\frac{\partial^2F}{\partial z^2}\right)\\ \sigma_\theta &= \frac{\partial}{\partial z}\left(2G\frac{\partial^2F}{\partial r^2}-\alpha D^2F+\beta\frac{\partial^2F}{\partial z^2}\right), \qquad \tau_{\theta z} = G_1\frac{\partial^2\varphi}{\partial r\,\partial z}\\ \sigma_z &= \frac{\partial}{\partial z}\left(\alpha_1D^2F+\beta_1\frac{\partial^2F}{\partial z^2}\right), \qquad \tau_{r\theta} = GD_1^2\varphi \end{aligned} \tag{1.5}$$

The functions F and ϕ satisfy the equations

$$\left(\frac{\partial^2}{\partial z^2}+s_1^2D^2\right)\left(\frac{\partial^2}{\partial z^2}+s_2^2D^2\right)F = 0, \qquad \left(\frac{\partial^2}{\partial z^2}+s_0^2D^2\right)\varphi = 0 \tag{1.6}$$

The first function F defines the axisymmetric deformation, the second defines the torsion. Evidently, s_0 is always a real number, and it is shown in [2] that s_1 and s_2 cannot be purely imaginary. The function F differs from that given in [2] by a constant multiplier. Hu Hai-chang has shown in [3] that in the general case of the deformation of a transversely isotropic body the stresses and displacements can be expressed in terms of two functions F and ϕ which satisfy Equations (1.6), where

$$D^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{1.7}$$

In the future we shall assume that $s_1 \neq s_2$; the solutions when the values of s are equal can be found by a limiting process.

In order to derive solutions to the problems set we shall need to make use of the expressions

$$F = \sum_{k=0}^{\infty} F_k(r)\,z^k, \qquad \varphi = \sum_{k=0}^{\infty}\varphi_k(r)\,z^k \tag{1.8}$$

From the requirement that (1.8) must satisfy Equations (1.6), we obtain recurrence differential equations relating F_k and ϕ_k respectively with different suffixes. We can express the final result in a compact form by introducing the operators used by Lur'e [1]:

$$\frac{1}{D}\sin szD = sz - \frac{s^3z^3}{3!}D^2 + \frac{s^5z^5}{5!}D^4 - \ldots$$
$$\cos szD = 1 - \frac{s^2z^2}{2!}D^2 + \frac{s^4z^4}{4!}D^4 - \ldots \qquad (1.9)$$

We then obtain

$$F = \sin s_1zD\cdot\frac{F_{10}}{D} + \cos s_1zD\cdot\frac{F_{11}}{s_1D^2} + \sin s_2zD\cdot\frac{F_{20}}{D} + \cos s_2zD\cdot\frac{F_{21}}{s_2D^2} \qquad (1.10)$$
$$\varphi = \cos s_0zD\cdot\varphi_0 + \sin s_0zD\cdot\frac{\varphi_1}{D}$$

(where, for convenience, operators and functions are separated by dots).

Here F_{10}, F_{20}, F_{11}, F_{21}, ϕ_0, ϕ_1 are unknown functions of the variable r; they must be determined in such a way that all conditions on the cylindrical surfaces are satisfied. Substituting these results in (1.4) and (1.5), we obtain the following expressions for the displacements and stresses:

$$u_r = \frac{\partial}{\partial r}\Big[s_1\Big(-\cos s_1zD\cdot F_{10} + \sin s_1zD\cdot\frac{F_{11}}{s_1D}\Big) + s_2\Big(-\cos s_2zD\cdot F_{20} + \sin s_2zD\cdot\frac{F_{21}}{s_2D}\Big)\Big]$$
$$w = D^2\Big[(\gamma - \delta s_1^2)\Big(\sin s_1zD\cdot\frac{F_{10}}{D} + \cos s_1zD\cdot\frac{F_{11}}{s_1D^2}\Big) + (\gamma - \delta s_2^2)\Big(\sin s_2zD\cdot\frac{F_{20}}{D} + \cos s_2zD\cdot\frac{F_{21}}{s_2D^2}\Big)\Big]$$
$$u_\theta = \frac{\partial}{\partial r}\Big(\cos s_0zD\cdot\varphi_0 + \sin s_0zD\cdot\frac{\varphi_1}{D}\Big) \qquad (1.11)$$

$$\sigma_r = s_1\Big[2G\frac{1}{r}\frac{\partial}{\partial r} - (\alpha + \beta s_1^2)D^2\Big]\Big(\cos s_1zD\cdot F_{10} - \sin s_1zD\cdot\frac{F_{11}}{D}\Big) + s_2\Big[2G\frac{1}{r}\frac{\partial}{\partial r} - (\alpha + \beta s_2^2)D^2\Big]\Big(\cos s_2zD\cdot F_{20} - \sin s_2zD\cdot\frac{F_{21}}{D}\Big)$$
$$\sigma_z = D^2\Big[s_1(\alpha_1 - \beta_1s_1^2)\Big(\cos s_1zD\cdot F_{10} - \sin s_1zD\cdot\frac{F_{11}}{s_1D}\Big) + s_2(\alpha_1 - \beta_1s_2^2)\Big(\cos s_2zD\cdot F_{20} - \sin s_2zD\cdot\frac{F_{21}}{s_2D}\Big)\Big]$$
$$\tau_{rz} = \frac{\partial}{\partial r}D^2\Big[(\alpha + \beta s_1^2)\Big(\sin s_1zD\cdot\frac{F_{10}}{D} + \cos s_1zD\cdot\frac{F_{10}}{s_1D^2}\Big) + (\alpha + \beta s_2^2)\Big(\sin s_2zD\cdot\frac{F_{20}}{D} + \cos s_2zD\cdot\frac{F_{21}}{s_2D^2}\Big)\Big]$$
$$\tau_{\theta z} = s_0G_1\frac{\partial}{\partial r}(-\sin s_0zD\cdot D\varphi_0 + \cos s_0zD\cdot\varphi_1)$$
$$\tau_{r\theta} = GD_1^2\Big(\cos s_0zD\cdot\varphi_0 + \sin s_0zD\cdot\frac{\varphi_1}{D}\Big) \qquad (1.12)$$

The expression for σ_θ can be obtained from σ_r by replacing $\partial/r\partial r$ by $\partial^2/\partial r^2$.

2. Axisymmetric deformation. Suppose that on the cylindrical surfaces we are given pressures p and q, which have rotational symmetry and which vary according to a law of integer polynomials in z, t being zero. It will be sufficient to consider the case when each of the pressures is proportional to z^k, where k is an arbitrary integer; we can find solutions for loads given in the form of polynomials by means of superposition.

We shall consider first the case when the normal pressure is proportional to an even power of z, and when the tangential pressure is proportional to an odd power. The boundary conditions are

$$\sigma_r = p_{2m}\left(\frac{z}{l}\right)^{2m}, \qquad \tau_{rz} = q_{2m-1}\left(\frac{z}{l}\right)^{2m-1}, \qquad \tau_{r\theta} = 0 \qquad \text{at} \qquad r = b \tag{2.1}$$

$$\sigma_r = p_{2m}'\left(\frac{z}{l}\right)^{2m}, \qquad \tau_{rz} = q_{2m-1}'\left(\frac{z}{l}\right)^{2m-1}, \qquad \tau_{r\theta} = 0 \qquad \text{at} \qquad r = a$$

In Formulas (1.11) and (1.12) we must set $\phi_0 = \phi_1 = F_{11} = F_{21} = 0$; then $u_\theta = \tau_{\theta z} = \tau_{r\theta} = 0$. The expressions for σ_r and τ_{rz} in expanded form are as follows:

$$\begin{aligned}\sigma_r = {} & 2G\frac{1}{r}\frac{d}{dr}(s_1F_{10} + s_2F_{20}) - D^2\left[s_1(\alpha + \beta s_1^2)F_{10} + s_2(\alpha + \beta s_2^2)F_{20}\right] - \\ & - \frac{z^2}{2!}\Big\{2G\frac{1}{r}\frac{d}{dr}D^2(s_1^3F_{10} + s_2^3F_{20}) - D^4\left[s_1^3(\alpha + \beta s_1^2)F_{10} + \right. \\ & \left. + s_2^3(\alpha + \beta s_2^2)F_{20}\right]\Big\} + \ldots + (-1)^k\frac{z^{2k}}{(2k)!}\Big\{2G\frac{1}{r}\frac{d}{dr}D^{2k}(s_1^{2k+1}F_{10} + s_2^{2k+1}F_{20}) - \\ & - D^{2k+2}\left[s_1^{2k+1}(\alpha + \beta s_1^2)F_{10} + s_2^{2k+1}(\alpha + \beta s_2^2)F_{20}\right]\Big\} + \ldots\end{aligned} \tag{2.2}$$

$$\begin{aligned}\tau_{rz} = {} & z\frac{d}{dr}D^2\left[s_1(\alpha + \beta s_1^2)F_{10} + s_2(\alpha + \beta s_2^2)F_{20}\right] - \\ & - \frac{z^3}{3!}\frac{d}{dr}D^4\left[s_1^3(\alpha + \beta s_1^2)F_{10} + s_2^3(\alpha + \beta s_2^2)F_{20}\right] + \ldots \\ & \ldots + (-1)^{k-1}\frac{z^{2k-1}}{(2k-1)!}\frac{d}{dr}D^{2k}\left[s_1^{2k-1}(\alpha + \beta s_1^2)F_{10} + s_2^{2k-1}(\alpha + \beta s_2^2)F_{20}\right] + \ldots\end{aligned}$$

From conditions (2.1) the last powers of z in (2.2) will be $2m$ and $2m - 1$. We must therefore set

$$D^{2m+2}F_{10} = A_{2m+2}, \qquad D^{2m+2}F_{20} = C_{2m+2} \tag{2.3}$$

(where A, C are arbitrary constants). Whence, taking into account the structure of the operator D^2, we find successively by integration

$$D^{2m}F_{10} = \frac{A_{2m+2}}{4} r^2 + B_{2m} \ln r + A_{2m}$$

$$D^{2m-2}F_{10} = \frac{A_{2m+2}}{(2^2 2!)^2} r^4 + \frac{B_{2m}}{4} r^2 (\ln r - 1) + \frac{A_{2m}}{4} r^2 + B_{2m-2} \ln r + A_{2m-2}$$

$$\begin{aligned} D^{2m-2k}F_{10} = & \frac{A_{2m+2}}{[2^{k+1}(k+1)!]^2} r^{2k+2} + \frac{B_{2m}}{(2^k k!)^2} r^{2k} \left(\ln r - \frac{b_2^{k+1}}{k!}\right) + \\ & + \frac{A_{2m}}{(2^k k!)^2} r^{2k} + \frac{B_{2m-2}}{[2^{k-1}(k-1)!]^2} r^{2k-2} \left[\ln r - \frac{b_2^k}{(k-1)!}\right] + \\ & + \frac{A_{2m-2}}{[2^{k-1}(k-1)!]^2} r^{2k-2} + \ldots + \frac{B_{2m-2k+2}}{4} r^2 (\ln r - 1) + \\ & + \frac{A_{2m-2k+2}}{4} r^2 + B_{2m-2k} \ln r + A_{2m-2k} \qquad (k = 0, 1, 2, \ldots, m) \end{aligned} \tag{2.4}$$

Here A_i, B_i are arbitrary constants, b_2^{k+1} are Stirling's numbers [4]

$$b_2^{k+1} = k!\left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right) \tag{2.5}$$

For $k = m - 1$ we obtain D^2F_{10}, and for $k = m$, $D^0F_{10} = F_{10}$. The expressions for the operators on the function F_{20} have, of course, the same structure as (2.4), except that A_i, B_i are replaced by different constants which we shall denote by C_i and D_i.

Setting $r = b$ and $r = a$ in Formulas (2.2), and equating coefficients of z^{2m} and z^{2m-1} to the given quantities (see (2.1)), we obtain the equations for the coefficients

$$\begin{aligned} & (G - \alpha - \beta s_1^2) s_1^{2m+1} A_{2m+2} + (G - \alpha - \beta s_2^2) s_2^{2m+1} C_{2m+2} + \\ & \quad + \frac{2G}{b^2} (s_1^{2m+1} B_{2m} + s_2^{2m+1} D_{2m}) = (-1)^m (2m)! \frac{p_{2m}}{l^{2m}} \\ & (G - \alpha - \beta s_1^2) s_1^{2m+1} A_{2m+2} + (G - \alpha - \beta s_2^2) s_2^{2m+1} C_{2m+2} + \\ & \quad + \frac{2G}{a^2} (s_1^{2m+1} B_{2m} + s_2^{2m+1} D_{2m}) = (-1)^m (2m)! \frac{p'_{2m}}{l^{2m}} \\ & [(\alpha + \beta s_1^2) s_1^{2m-1} A_{2m+2} + (\alpha + \beta s_2^2) s_2^{2m-1} C_{2m+2}] \frac{b^2}{2} + [(\alpha + \beta s_1^2) s_1^{2m-1} B_{2m} + \\ & \quad + (\alpha + \beta s_2^2) s_2^{2m-1} D_{2m}] = (-1)^{m-1} (2m-1)! \frac{q_{2m-1} b}{l^{2m-1}} \\ & [(\alpha + \beta s_1^2) s_1^{2m-1} A_{2m+2} + (\alpha + \beta s_2^2) s_2^{2m-1} C_{2m+2}] \frac{a^2}{2} + [(\alpha + \beta s_1^2) s_1^{2m-1} B_{2m} + \\ & \quad + (\alpha + \beta s_2^2) s_2^{2m-1} D_{2m}] = (-1)^{m-1} (2m-1)! \frac{q'_{2m-1} a}{l^{2m-1}} \end{aligned} \tag{2.6}$$

Setting $r = b$ and $r = a$ and equating to zero the coefficients of z^{2m-2} in the expression for σ_r and those of z^{2m-3} in the expression for τ_{rz}, we obtain four equations which contain the constants A_{2m}, C_{2m}, B_{2m-2}, D_{2m-2} in addition to those already found. Proceeding in this way from higher to lower powers of z, we eventually arrive at a term in the expression for σ_r which is independent of z, and in this way we obtain a set of two equations for two linear combinations of coefficients

$$(G - \alpha - \beta s_1^2)\, s_1 A_2 + (G - \alpha - \beta s_2^2)\, s_2 C_2, \qquad s_1 B_0 + s_2 D_0 \tag{2.7}$$

from which the coefficients can be determined uniquely.

As a result we can determine u_r, σ_r, σ_θ, τ_{rz} accurately, but an arbitrary constant (which we can take as A_2 or C_2 or any linear combination of these constants) will occur in the expressions for w and σ_z. The arbitrary constant can be found by equating the vector sum of the forces on one end of the cylinder to the given value, in particular to zero; it is not difficult to show that the forces on the other end of the cylinder then balance the given external loading. The end section $z = 0$ remains plane.

If the normal pressure is proportional to an odd power of z and the tangential pressure is proportional to an even number, then we have the conditions

$$\begin{aligned} \sigma_r &= p_{2m+1}\left(\frac{z}{l}\right)^{2m+1}, \quad \tau_{rz} = q_{2m}\left(\frac{z}{l}\right)^{2m}, \quad \tau_{r\theta} = 0 \quad \text{at } r = b \\ \sigma_r &= p_{2m+1}'\left(\frac{z}{l}\right)^{2m+1}, \quad \tau_{rz} = q_{2m}'\left(\frac{z}{l}\right)^{2m}, \quad \tau_{r\theta} = 0 \quad \text{at } r = a \end{aligned} \tag{2.8}$$

This problem can be solved in a completely analogous way. We equate to zero the functions ϕ_0, ϕ_1, F_{10}, F_{20} and we obtain the same expressions (2.4) for $D^{2m-2k}F_{11}$, $D^{2m-2k}F_{21}$.

In this case all the coefficients except A_0, C_0 can be found from the boundary conditions (2.8), and the expressions for the stresses contain no arbitrary constants. On the end $z = 0$ the stress σ_z is zero, and on the other end $z = l$ it balances the external pressure q.

It is not difficult to show, by making use of Expressions (1.3), that when s_1 and s_2 are not equal the determinants of all the sets of equations are nonzero.

For a solid cylinder ($a = 0$) the above formulas can be simplified considerably, since all terms in (2.4) containing logarithms must be discarded, since they lead to singularity at $r = 0$.

Example. **A hollow cylinder subjected to a normal pressure distributed**

over its external surface according to a parabolic law (see Figure).

We have: $m = 1$

$$p = -n_2\left(\frac{z}{l}\right)^2, \qquad q = p' = q' = 0, \qquad \varphi_0 = \varphi_1 = F_{11} = F_{21} = 0$$

$$D^4 F_{10} = A_4, \qquad D^4 F_{20} = C_4$$

$$D^2 F_{10} = \frac{A_4}{4} r^2 + B_2 \ln r + A_2, \qquad D^2 F_{20} = \frac{C_4}{4} r^2 + D_2 \ln r + C_2 \tag{2.9}$$

$$F_{10} = \frac{A_4}{64} r^4 + \frac{B_2}{4} r^2 (\ln r - 1) + \frac{A_2}{4} r^2 + B_0 \ln r + A_0$$

$$F_{20} = \frac{C_4}{64} r^4 + \frac{D_2}{4} r^2 (\ln r - 1) + \frac{C_2}{4} r^2 + D_0 \ln r + C_0$$

After finding the constants from the boundary conditions, we finally obtain the following expressions for the stresses and displacements:

$$\sigma_r = -\frac{n_2 b^2}{(b^2 - a^2)\, l^2}\left\{ z^2\left(1 - \frac{a^2}{r^2}\right) + \frac{\nu_1}{1 - \nu_1\nu_2}\left[\frac{1-\nu}{4}\left(b^2 + a^2 - r^2 - \frac{a^2 b^2}{r^2}\right) + \right.\right.$$
$$\left.\left. + (1+\nu)\, a^2\left(\frac{b^2 \ln b - a^2 \ln a}{b^2 - a^2} - \frac{a^2 b^2 \ln b/a}{b^2 - a^2}\cdot\frac{1}{r^2} - \ln r\right)\right]\right\}$$

$$\sigma_\theta = -\frac{n_2 b^2}{(b^2 - a^2)\, l^2}\left\{ z^2\left(1 + \frac{a^2}{r^2}\right) + \frac{\nu_1}{1 - \nu_1\nu_2}\left[\frac{1-\nu}{4}\left(b^2 + a^2 - 3r^2 + \frac{a^2 b^2}{r^2}\right) + \right.\right.$$
$$\left.\left. + (1+\nu)\, a^2\left(\frac{b^2 \ln b - a^2 \ln a}{b^2 - a^2} + \frac{a^2 b^2 \ln b/a}{b^2 - a^2}\cdot\frac{1}{r^2} - \ln r - 1\right)\right]\right\} \tag{2.10}$$

$$\sigma_z = \frac{n_2 b^2}{(b^2 - a^2)\, l^2}\,\frac{E_1}{E\,(1 - \nu_1\nu_2)}\left[(1-\nu) r^2 + (1+\nu) a^2 + 2(1+\nu)\, a^2\left(\ln r - \right.\right.$$
$$\left.\left. - \frac{b^2 \ln b - a^2 \ln a}{b^2 - a^2}\right) + C\,(1 - \nu_1\nu_2)\right]$$

$$\tau_{rz} = 0$$

$$u_r = -\frac{n_2 b^2}{E\,(b^2 - a^2) l^2}\left\{ z^2\left[(1-\nu) r + (1+\nu)\frac{a^2}{r}\right] + \frac{\nu_1}{1 - \nu_1\nu_2}\,\frac{1-\nu}{4}\left[(1-\nu)(b^2 + a^2)\, r - \right.\right.$$
$$\left. - (3-\nu)\, r^3 + (1+\nu)\frac{a^2 b^2}{r}\right] + \frac{\nu_1 (1+\nu)}{1 - \nu_1\nu_2}\, a^2\left[(1-\nu)\,\frac{b^2 \ln b - a^2 \ln a}{b^2 - a^2}\, r + (1+\nu) \times \right.$$
$$\left.\left. \times \frac{a^2 b^2 \ln b/a}{b^2 - a^2}\,\frac{1}{r} - (1-\nu)\, r \ln r - r\right]\right\} \tag{2.11}$$

$$w = \frac{n_2 b^2}{E\,(b^2 - a^2)\, l^2}\left\{\frac{2\nu_2}{3} z^3 + z\left[(1-\nu) r^2 + 2(1+\nu)\, a^2 \ln r + C\right]\right\}, \qquad u_\theta = 0$$

The arbitrary constant C is found from the conditions on the ends; if there is no externally applied pressure here, we obtain

$$C = -0.5\,\lambda\,(b^2 + a^2) \qquad \left(\lambda = \frac{1-\nu}{1 - \nu_1\nu_2}\right) \tag{2.12}$$

For a solid cylinder we find from (2.10) and (2.11) that

$$\sigma_r = -\frac{n_2}{l^2}\left[z^2 + \frac{\lambda\nu_1}{4}(b^2 - r^2)\right], \quad \sigma_\theta = -\frac{n_2}{l^2}\left[z^2 + \frac{\lambda\nu_1}{4}(b^2 - 3r^2)\right], \quad \sigma_z = \frac{n_2}{l^2}\frac{E_1}{E}(\lambda r^2 + C) \tag{2.13}$$

3. Torsion. Suppose that on the surfaces $r = b$, $r = a$ of a hollow cylinder only the pressures t are applied (see Figure) which are independent of θ and which are distributed according to a polynomial law. One end will be assumed to be free from load and the other will be assumed to be fixed.

Here again it is sufficient to consider the case of a load proportional to some power of z. Let us suppose that this is an even number. We have the boundary conditions

$$\begin{aligned} \sigma_r &= \tau_{rz} = 0, \quad \tau_{r\theta} = t_{2m}\left(\frac{z}{l}\right)^{2m} \quad \text{at } r = b \\ \sigma_r &= \tau_{rz} = 0, \quad \tau_{r\theta} = t_{2m}'\left(\frac{z}{l}\right)^{2m} \quad \text{at } r = a \end{aligned} \tag{3.1}$$

In Formulas (1.11) and (1.12) we must set

$$F_{10} = F_{20} = F_{11} = F_{21} = \varphi_1 = 0, \qquad \sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0$$

The expressions for the displacements u_θ and the stresses $\tau_{r\theta}$ in expanded form can be written as follows:

$$u_\theta = \frac{d}{dr}\left[\varphi_0 - \frac{s_0^2 z^2}{2!} D^2\varphi_0 + \ldots + (-1)^k \frac{s_0^{2k} z^{2k}}{(2k)!} D^{2k}\varphi_0 + \ldots\right] \tag{3.2}$$

$$\tau_{r\theta} = G D_1^2\left[\varphi_0 - \frac{s_0^2 z^2}{2!} D^2\varphi_0 + \ldots + (-1)^k \frac{s_0^{2k} z^{2k}}{(2k)!} D^{2k}\varphi_0 + \ldots\right] \tag{3.3}$$

If we discard powers of z higher than $2m$ in the expression for $\tau_{r\theta}$, we obtain the following equation for ϕ_0:

$$D_1^2 D^{2m+2}\varphi_0 = 0 \tag{3.4}$$

Whence

$$\begin{aligned} D^{2m+2}\varphi_0 &= \frac{A_{2m+4}}{4} r^2 + A_{2m+2} \\ D^{2m}\varphi_0 &= \frac{A_{2m+4}}{(2^2 2!)^2} r^4 + \frac{A_{2m+2}}{4} r^2 + B_{2m}\ln r + A_{2m} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D^{2m-2k}\varphi_0 &= \frac{A_{2m+4}}{[2^{k+2}(k+2)!]^2} r^{2k+4} + \frac{A_{2m+2}}{[2^{k+1}(k+1)!]^2} r^{2k+2} + \\ &+ \frac{B_{2m}}{(2^k k!)^2} r^{2k}\left(\ln r - \frac{b_2^{k+1}}{k!}\right) + \frac{A_{2m}}{(2^k k!)^2} r^{2k} + \ldots \end{aligned} \tag{3.5}$$

$$\ldots + \frac{B_{2m-2k+2}}{4} r^2 (\ln r - 1) + \frac{A_{2m-2k+2}}{4} r^2 + B_{2m-2k} \ln r + A_{2m-2k}$$

$$D_1{}^2 D^{2m-2k} \varphi_0 = \frac{A_{2m+4}}{2^{2k+2} k!\,(k+2)!} r^{2k+2} + \frac{A_{2m+2}}{2^{2k}(k-1)!\,(k+1)!} r^{2k} +$$

$$+ \frac{B_{2m}}{2^{2k-2}(k-2)!\,k!} r^{2k-2} \left[\ln r + \frac{2k-1}{2k(k-1)} - \frac{b_2{}^{k+1}}{k!} \right] + \frac{A_{2m}}{2^{2k-2}(k-2)!\,k!} r^{2k-2} + \ldots$$

$$\ldots + \frac{B_{2m-2k+4}}{8} r^2 \left(\ln r - \frac{3}{4} \right) + \frac{A_{2m-2k+4}}{8} r^2 + \frac{B_{2m-2k+2}}{2} - \frac{2B_{2m-2k}}{r^2} \quad (3.6)$$

$$(k = 0, 1, 2, \ldots, m)$$

If we now satisfy the conditions (3.1) we obtain the equations

$$(D_1{}^2 D^{2m} \varphi_0)_{r=b} = \frac{t_{2m}}{l^{2m}} \quad (D_1{}^2 D^{2m} \varphi_0)_{r=a} = \frac{t_{2m}'}{l^{2m}} \quad (3.7)$$

$$D_1{}^2 D^{2m-2} \varphi_0 = 0, \quad D_1{}^2 D^{2m-4} \varphi_0 = 0, \quad \ldots, \quad D_1{}^2 \varphi_0 = 0 \quad \text{at } r = b \text{ and } r = a$$

From these equations we determine successively the constants A_{2m+4}, B_{2m}; A_{2m+2}, B_{2m-2}; ...; A_6, B_2; A_4, B_0. The coefficient A_0 does not appear in the formulas, and A_2 appears only in the expression for the displacement; we determine A_2 from the requirement that some circle $r = R$ in the plane of the fixed end (for example, the external or internal contour of the section) is not displaced. On the free end $\tau_{\theta z} = 0$; on the fixed end this stress balances the external pressure.

If the stress t is proportional to an odd power of z the problem can be solved in a completely analogous way. In Formulas (1.11) and (1.12) we must equate to zero all functions except ϕ_1. In the expression for $\tau_{\theta z}$ there will appear an arbitrary constant which must be found from the requirement that the torque on the free end is zero.

Example. A hollow cylinder subjected to torques applied over the external surface according to a parabolic law.

We have $m = 1$

$$t = t_2 \left(\frac{z}{e} \right)^2, \qquad t' = 0, \qquad \varphi_1 = 0$$

$$D_1{}^2 D^4 \varphi_0 = 0$$

$$D_1{}^2 D^2 \varphi_0 = \frac{A_6}{8} r^2 - \frac{2B_2}{r^2} \quad (3.8)$$

$$D_1{}^2 \varphi_0 = \frac{A_6}{96} r^4 + \frac{A_4}{8} r^2 + \frac{B_2}{2} - \frac{2B_0}{r_2}$$

Having determined the four constants from Equations (3.7), we can find the expressions for the stresses

$$\tau_{r\theta} = \frac{t_2 b^2}{(b^4 - a^4)\, l^2} \left\{ z^2 \left(r^2 - \frac{a^4}{r^2} \right) + \frac{G_1}{6G\,(b^2 + a^2)} \left[(b^4 + a^2 b^2 + 4a^4)\, r^2 + \frac{a^4 b^2 (3a^2 - b^2)}{r^2} - \right.\right.$$
$$\left.\left. - (b^2 + a^2)(r^4 + 3a^4) \right] \right\} \qquad (3.9)$$
$$\tau_{\theta z} = \frac{t_2 b^2}{(b^4 - a^4)\, l^2} \left[-\frac{4}{3} z^3 r + \frac{G_1}{G} z \left(r^3 + \frac{a^4}{r} - \frac{2}{3} \frac{b^4 + a^2 b^2 + 4a^4}{b^2 + a^2} r \right) \right]$$

As we might have expected, this result coincides with that obtained by another method [5].

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